Turbulent entrainment in stratified flows

By T. H. ELLISON AND J. S. TURNER
Department of the Mechanics of Fluids, University of Manchester

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When a fluid which is lighter than its surroundings is emitted by a source under a sloping roof (or a heavier fluid from a source on a sloping floor), it may flow as a relatively thin turbulent layer. The motion of this layer is governed by the rate at which it entrains the ambient fluid. A theory is presented in which it is assumed that the entrainment is proportional to the velocity of the layer multiplied by an empirical function, $E(\mathrm{Ri})$, of the overall Richardson number for the layer defined by $\mathrm{Ri} = g(\rho_a - \rho)\, h/\rho_a V^2$. This theory predicts that in most practical cases the layer will rapidly attain an equilibrium state in which Ri does not vary with distance downstream, and the gravitational force on the layer is just balanced by the drag due to entrainment together with friction on the floor or roof.

Two series of laboratory experiments are described from which $E(\mathrm{Ri})$ can be determined. In the first, the spread of a surface jet of fluid lighter than that over which it is flowing is measured; in the second, a study is made of the flow of a heavy liquid down the sloping floor of a channel. These experiments show that E falls off rapidly as Ri increases and is probably negligible when Ri is more than about 0·8.

The theoretical and experimental results allow predictions to be made of flow velocities once the rate of supply of density difference is known. An estimate is also given of the uniform velocity which the ambient fluid must possess in order to cause the motion of the layer to be reversed.

1. Introduction

There are many situations in nature in which a thin layer of fluid lighter than its surroundings flows up under a sloping roof, or a layer of fluid heavier than its surroundings flows down a sloping floor. In some cases such flows are of considerable economic importance. For example, katabatic winds in the atmosphere, in which air cooled by contact with cold ground flows downhill, are of importance to agriculture because of their influence on frost damage; also they may carry smoke or fog and so affect visibility at airfields and ports. The accumulation of smog in valleys is a slightly different but related problem (Scorer 1954). In some regions, such as the coasts of Greenland and Antarctica, katabatic winds may attain great strength (Ball 1956). Some bottom currents in the ocean are similar; Dietrich (1956) has described the flow of cold water from the Arctic over a submarine ridge between Iceland and the Faroes. In estuaries, where lighter river water meets the sea, a variety of complicated phenomena may occur (Stommel 1953); but in certain fjord-like estuaries, such as the Alberni Inlet (Tully 1949), the fresh water flows out on the sea surface in a manner which closely resembles the experiments to be described in § 5.

Heavy layers may also be formed on account of the extra weight of suspended solids. This is the case in turbidity currents in the oceans (Kuenen 1952) and probably to some extent in dust-storms in the atmosphere such as the Sudan haboob (Freeman 1952). In these cases, and also in connexion with some phenomena of sea-breezes, interest centres more on the rate of advance of the leading edge of the heavy fluid than on the steady state (Berson 1958) though the latter is still important. Density currents in reservoirs (Bell 1942) are of interest to engineers because of their effect on silting. The most important example of a flow in which the moving layer is lighter than its surroundings is to be found in coal mines. Methane can accumulate near the roof and the way in which it flows and the extent to which it mixes with air have an important bearing on safety. It was this aspect of the problem which led to the work reported here being undertaken.

When planning an investigation of the flows listed above, it is useful to realize that the range of conditions which may be encountered in one context, such as the flow of methane in mines, may be greater than the difference between that situation and an apparently disconnected one, such as the flow in an estuary. It is therefore desirable to classify the flows according to fundamental hydrodynamical properties and not geophysical context. One obvious group of such properties is the relative depth, velocity and level of turbulence in the thin layers and the ambient fluid. In order to illustrate this, let us consider a source of methane in the roof of a sloping gallery in a mine. If the ambient air is at rest, the methane will form a thin layer and move up along the roof; the thickness and velocity of the layer will depend on the slope and the rate of emission of methane. If the velocity is small enough the layer may remain laminar for a considerable distance; otherwise it will become turbulent. If we now add the possibility that the ambient air may itself be moving and may either be effectively laminar (in most practical cases a very low level of turbulence would be implied rather than true laminar flow) or highly turbulent, it will be seen that a large number of cases must be considered. Each of these is likely to show its own phenomena and require its own theory.

This paper is concerned only with the case when the thin layer is turbulent and the ambient fluid either at rest or moving with a very low level of turbulence. The situation has similarities both with the motion of a buoyant plume and with the ordinary flow of water in an open channel. A theory is presented, describing the overall properties of the layer, which is based on the same physical principles as the theory of plumes but takes account of the stabilizing effect of the density difference across the edge of the layer. In place of a single entrainment constant, an entrainment function, $E(\mathrm{Ri})$, depending on the overall Richardson number of the layer, is used. This function has to be determined empirically. The theory leads to a generalization of the equations for slowly varied flow in hydraulics.

Two sets of experiments are reported which enable the function $E(\mathrm{Ri})$ to be deduced and the profiles of velocity and density in the layers to be studied. From these it is seen that stability will normally have a drastic effect on entrainment in many natural situations. While the experiments give general support for the theory, they reveal a number of minor complications.

In the final part of this paper, the application of the work is discussed and the effect of movement of the ambient fluid is calculated.

2. Entrainment

The concept of entrainment

Plumes and jets are examples of free turbulent flows; the region of turbulence is of finite extent and adjoins non-turbulent fluid of almost uniform mean velocity. The turbulent region grows with distance downstream as the non-turbulent fluid becomes *entrained* in it. This entrainment implies a flow of ambient fluid into the turbulent layer, and so, in the case where the ambient fluid is initially at rest, a small mean velocity perpendicular to the main flow.

Although the distribution of stress is not exactly the same in plumes as in jets, it is known that the velocity profiles are roughly Gaussian in both cases, and it is likely that the large-scale features of the turbulence are also much the same. It is also probable that the turbulence is not far from a state of local equilibrium uninfluenced by upstream history; and if so it must be determined by local scales of velocity and length.

Provided the Reynolds number is high enough, we should not expect molecular viscosity to influence the flow greatly (Townsend 1956). There is just a chance that the Prandtl (or Schmidt) number, ν/D, may influence the turbulent diffusion of the agent causing the density variation (temperature, salt concentration, etc.) even at very high Reynolds numbers. This was suggested by Batchelor & Townsend (1956); but since they were unable to predict the magnitude of the effect, their ideas have not been widely accepted. The point is an important one, since it bears on the relevance of experiments in a water channel to large-scale situations where the fluid is a gas, but in the absence of further information we can only ignore it for the present with a note of caution. If this is done, the velocity of inflow into the turbulent region must be proportional to the velocity scale of the layer; the constant of proportionality is called the *entrainment constant*, E.

The idea of an entrainment constant was first used explicitly by Morton, Taylor & Turner (1956) who made it the basis for their theory of plumes. More recently Morton (1959) has extended the theory to *forced plumes*, which are emitted with high momentum and so behave initially as jets. In all these cases the boundaries of the turbulent region have been not far from vertical, so it has been unnecessary to consider the action of a stable density gradient in inhibiting mixing; we refer to them as neutral cases. In the cases with which this paper is particularly concerned, such stability has a drastic influence on the value of the entrainment constant.

The numerical value of E clearly depends on the definition of the length scale that is used, and the use of scales associated with particular shapes of profile has led to some confusion in the past. Our presentation is in terms of integrals over the profiles and avoids any assumption about their shape. For brevity, we consider only the two-dimensional case.

Definition of the entrainment constant

Take the x-axis along the plume or jet, and the y-axis perpendicular to it, measuring y from the plane of symmetry when both sides of the flow are free and from the boundary when the plume is along a floor or roof. Let $v(x, y)$ denote the mean velocity relative to that in the ambient fluid, and let $V(x)$, $h(x)$ and $\Delta(x)/g$ be the velocity, length and density scales in the cross-section, and A the flux of density difference (which is independent of x). Then the quantities V, h, A and Δ are defined by the relations:

$$Vh = \int v\,dy, \tag{1}$$

$$V^2h = \int v^2\,dy, \tag{2}$$

$$\begin{aligned} A &= \int \left(\frac{\rho_a - \rho}{\rho_a}\right) g(v + V_a)\,dy \\ &= (V + V_a)h\Delta, \end{aligned} \tag{3}$$

where the suffix a refers to the ambient fluid.

The entrainment constant, E, can now be defined in accordance with the qualitative definition given earlier by

$$\frac{dh(V + V_a)}{dx} = E\,|V|. \tag{4}$$

Entirely analogous definitions can be given in the case of a circularly symmetrical plume or jet.

Evaluation of $E(0)$ for plumes and jets

Values of E for the neutral case can be determined from published measurements of plumes and jets.

For the two-dimensional jet (Townsend 1956, p. 193), $V \propto x^{\frac{1}{2}}$, $h = 0{\cdot}15x$, so $E = 0{\cdot}075$. For the circular jet (Townsend 1956, p. 186) we find $E = 0{\cdot}070$.

Measurements of plumes are notoriously unreliable, since the velocities involved are small and only a very little movement in the ambient fluid is needed to cause a waving of the plume and so a spurious spreading. Rouse, Yih & Humphreys (1952) have given measurements of the spread of circular and two-dimensional plumes. In the circular case their measurements lead to $E = 0{\cdot}12$ and in the two-dimensional case to $E = 0{\cdot}22$.

Although these are probably the best available measurements of plumes, they cannot be regarded as definitive. In the two-dimensional case, the source was surrounded by a wall which may have affected the flow. Measurements to be presented later in this paper do in fact confirm that the value found for two-dimensional plumes is far too high.

Another method of estimating the entrainment constant for plumes was used by Morton *et al.* (1956). They observed the height to which a plume would rise in a stably stratified environment. Allowing for the different definition used in their paper, they also found E to be 0·12 for circular plumes. Here again the measurements cannot be considered to be very precise; in particular, the greatest widths of their plumes were not small compared with the width of the tank and there must have been some downflow in the ambient fluid.

To sum up, we may say that there is some suggestion that E is greater in plumes than in jets, but that measurements on plumes have not been sufficiently precise to settle the point.

Effect of a boundary

Bakke (1957) has published measurements of a wall-jet, in which the velocity profile resembles that of one half of an ordinary two-dimensional jet, modified by the presence of a thin boundary layer along the wall. His measurements lead to a value for E of 0·072. The slight extent of the modification caused by the presence of the wall suggests that it can be taken into account by including a frictional force in the momentum equation. The outer parts of the flow are little affected by the wall, and E is in excellent agreement with the values given earlier for jets.

We shall see below, however, that in the case of inclined plumes the situation is more complex. At the rather low Reynolds numbers attainable in the laboratory, there is a tendency for the region between the wall and the velocity maximum not to become fully turbulent, and this can influence the whole flow.

3. The effect of stable stratification

It is well known that a stable density gradient strongly inhibits turbulent mixing. Richardson (1920) showed that this was to be expected from consideration of the turbulent energy balance, and in recent work (Ellison 1957; Townsend 1958) further suggestions have been put forward about the mechanism whereby the buoyancy forces destroy the turbulence. Thus when the presence of a roof or floor causes a plume to be inclined so that the mixing zone is no longer nearly vertical as in the ordinary plume, the density difference between it and the ambient fluid will give rise to a change in the value of E.

If we assume a state of local equilibrium to obtain and ignore molecular effects, we see that the only relevant dimensionless groups upon which E can depend are the absolute density difference $(\rho-\rho_a)/\rho_a$, and some parameter in the form of a Richardson number specifying the stability of the layer. This we define by:

$$\mathrm{Ri} = \frac{\Delta h \cos\alpha}{V^2} = \frac{A \cos\alpha}{V^2(V+V_a)}, \tag{5}$$

where α is the slope of the floor. Note that the Richardson number used in this discussion is a parameter specifying the *overall* state of a cross-section of the layer; and it must not be confused with the local parameter based on gradients of density and velocity. Ri is equal to the inverse square of the internal Froude number for the layer, and the analysis could equally have been written in terms of Fr, which necessarily represents the same physical quantity as Ri.

Now it is generally agreed (Batchelor 1953) that provided the density difference is small enough, it can be neglected except where multiplied by g as in the Richardson number. This is equivalent to ignoring $(\rho-\rho_a)/\rho_a$ in the inertia terms in the equations of motion; in other words, the fluid is treated as having uniform mass but variable weight. We should note, however, that there are cases of practical importance, such as unmixed methane layers, where the density differences are large and it may not be justifiable to neglect this parameter.

Thus we may assert that provided the Reynolds number is sufficiently large and the density differences are small, E is a function only of the Richardson number, Ri. This assertion forms the basis of the theory presented in the next section. One should remember, however, that there are many situations in the laboratory and in nature when the Reynolds number is not sufficiently high to be unimportant; we shall discuss these further later.

4. The theory of inclined plumes

We consider the steady state of a layer moving up a roof over a stationary ambient fluid. The velocity and density profiles are left arbitrary, their shape being taken into account by the inclusion of two dimensionless coefficients S_1 and S_2 in the theory.

It is assumed that the ambient fluid is in hydrostatic equilibrium and that the velocities in the plume at right angles to the boundary are sufficiently small for the pressure to be found everywhere by integrating the hydrostatic equation in that direction.

The equations for continuity of mass deficiency and for entrainment can now be written as follows:

$$Vh\Delta = A = \text{constant}, \tag{6}$$

$$\frac{1}{V}\frac{d(Vh)}{dx} = E, \tag{7}$$

where E is a function of Ri, which in this case may be written $A\cos\alpha/V^3$. The momentum equation is

$$\frac{d(V^2h)}{dx} = -CV^2 - \frac{1}{2}\frac{d(S_1\Delta h^2\cos\alpha)}{dx} + S_2\Delta h\sin\alpha, \tag{8}$$

where
$$S_1\Delta h^2 = \int_0^\infty 2g\left(\frac{\rho_a-\rho}{\rho_a}\right)y\,dy \tag{9}$$

and
$$S_2\Delta h = \int_0^\infty g\left(\frac{\rho_a-\rho}{\rho_a}\right)dy. \tag{10}$$

In the momentum equation (8), we have made use of the fact that $(\rho_a-\rho)/\rho_a$ can be neglected except where it is combined with g. Equation (8) represents the momentum balance of the fluid contained in the elementary volume of unit width bounded by x, $x+dx$, $y = 0$, $y = \infty$. On the left-hand side is the rate of change of momentum of the fluid. The first term on the right is the frictional drag of the roof; the second expresses the pressure force on the layer due to its changing depth and density, that is, the difference of the integrated hydrostatic pressure on the planes x and $x+dx$; and the last term represents the force of gravity accelerating the layer. Note that while we have written the friction term as a force proportional to the square of velocity, the drag coefficient C will not be a constant even for a given wall; it will vary with the shape of the profiles and the local stability in the region near the wall. A similar effect is observed in the lower atmosphere where the drag coefficient referred to the wind at a fixed height decreases with increasing stability.

If any variation of S_1, S_2 and α with x is ignored, and use is made of (5), (8) can be written:

$$S_2 \mathrm{Ri} \tan\alpha - C = (1+\tfrac{1}{2}S_1\mathrm{Ri})\frac{dh}{dx} - \frac{(2-\tfrac{1}{2}S_1\mathrm{Ri})\,h}{3\mathrm{Ri}}\frac{d\mathrm{Ri}}{dx}. \tag{11}$$

It will be useful to have in mind the magnitudes of S_1 and S_2 during our theoretical discussion. In the experiments on inclined plumes to be described in § 5 we found values ranging from 0·2 to 0·3 for S_1 and from 0·6 to 0·9 for S_2. The terms in (11) involving S_1 are likely to be relatively unimportant.

The development of the layer

The entrainment equation (7) now enables us to derive from (11) separate equations for dh/dx and $d\mathrm{Ri}/dx$. These are:

$$\frac{dh}{dx} = \frac{(2-\tfrac{1}{2}S_1\mathrm{Ri})\,E - S_2\mathrm{Ri}\tan\alpha + C}{1-S_1\mathrm{Ri}}, \tag{12}$$

and

$$\frac{h}{3\mathrm{Ri}}\frac{d\mathrm{Ri}}{dx} = \frac{(1+\tfrac{1}{2}S_1\mathrm{Ri})\,E - S_2\mathrm{Ri}\tan\alpha + C}{1-S_1\mathrm{Ri}}. \tag{13}$$

These equations play a role in the theory of inclined plumes analogous to that played by Bresse's (1860) equation in the mathematical theory of gradually varying flow of a homogeneous liquid in an open channel; if one sets $E = 0$, they reduce to Bresse's equation.

In ordinary hydraulics h and Ri are directly connected through the equation of continuity (which is simply Vh = constant in that case) so there is a certain depth at which Ri = 1 and another depth for which $dh/dx = 0$ and $d\mathrm{Ri}/dx = 0$. In the case of the inclined plume, the behaviour of Ri is little different from that in ordinary hydraulics, but h is no longer connected uniquely with it. There is a particular value of Ri, called the *normal* value, Ri_n, determined by the slope and friction, for which the right-hand side of (13) vanishes and $d\mathrm{Ri}/dx = 0$; to determine Ri_n one must know S_1 and S_2 and use measured values for E as a function of Ri. It is found that in many practical flows Ri attains the value Ri_n in a short distance, afterwards remaining constant, but h continues to increase steadily; in fact when $\mathrm{Ri} = \mathrm{Ri}_n$,

$$\frac{dh}{dx} = E(\mathrm{Ri}_n). \tag{14}$$

The analogy with hydraulic theory may be pursued further. Equations (12) and (13) have six different forms of solution depending on the relative sizes of

S_1Ri at the origin, S_1Ri$_n$ and 1. The distinction can still be made between tranquil (subcritical) flow in which S_1Ri is greater than unity and the velocity of long internal waves on the layer is greater than the flow velocity, and shooting (supercritical) flow in which S_1Ri is less than unity and disturbances are unable to propagate upstream. It will appear that the entrainment becomes very small when Ri is still somewhat less than unity, so that all tranquil flows are similar to the corresponding flows in ordinary hydraulics. We shall avoid the use of the words 'subcritical' and 'supercritical' to describe the state of the flow, since there is risk of confusion with the idea of a critical Richardson number at which the entrainment effectively vanishes.

The type of solution of most concern to us is that for flow down steep slopes (discussed, for example, by Bakhmeteff (1932, pp. 75–81)) in which S_1Ri$_n$ and S_1Ri$_0$ are both less than unity. The flow is shooting throughout. If it starts too slowly, gravity accelerates the flow until Ri approaches Ri$_n$; if it starts too fast, increased mixing occurs until Ri$_n$ is again approached and there is a balance between gravity and bottom friction and entrainment.

A numerical solution of equations (12) and (13), using the knowledge of E(Ri) obtained in a later section of this paper, shows that if, for example, the layer was initiated with Ri $= 1$ and depth h_0 on a slope of 0·1, then at $x = 10h_0$, Ri would have attained a value within 0·01 of Ri$_n$. In fact, the rate of change of Ri near the origin is so rapid that the theory cannot be expected to be at all accurate; nevertheless the conclusion that the normal state is reached rapidly should be correct. This conclusion probably applies to a wide range of cases, and it is used in interpreting our experiments. It implies that, at a given slope, the velocity of an inclined plume depends on the rate of output of density deficiency, A, alone. The rate of spread of the edge of the plume and the entrainment E will not depend on the depth of the layer nor on the distance downstream.

5. The experimental determination of entrainment: the surface jet

The foregoing theoretical discussion leads us to expect the entrainment to depend principally on the Richardson number, but it is not yet possible to predict its magnitude or the form of the function E(Ri) on purely theoretical grounds; these must be found experimentally. In this and the following sections two sets of experiments are described. The first were begun before the ideas of this paper had been developed, and there are therefore several features of them which may be criticized in the light of our present knowledge; however, they gave the first indication of the shape of the E *vs* Ri relation and seem worth presenting for comparison with the later independent measurements, which relate more directly to the theory.

The surface jet

The early experiments were carried out while the second author was in Cambridge and were influenced by the equipment available in the Cavendish Laboratory. It was convenient to use a tank about 15 cm wide, 20 cm deep and 5 m long, with plate-glass sides and wooden bottom and ends. At first this was fitted with a sloping floor and attempts were made to study gravity flows directly. The essential difficulties will be discussed in connexion with the next set of experiments, but this attempt was abandoned in favour of a simpler experiment which permitted the mixing to be studied without the complications due to slope and friction. The tank was divided into two short end sections and a long centre compartment as shown in figure 1, by watertight partitions which were fastened to the sides and bottom but were slightly lower than the top of the tank. The experiment consisted of filling the centre section (*b*) with salt solution of a known density, and allowing a steady supply of tap water to fill (*a*) and flow over into (*b*) and thence to the tank (*c*), which was simply to catch this overflow.

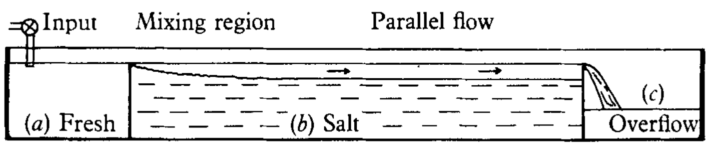

FIGURE 1. Sketch of channel used for the surface jet experiments.

With no density difference between (*a*) and (*b*) the flow formed a simple two-dimensional half-jet which at sufficiently high rates of flow became fully turbulent after a few centimetres. This then increased its depth nearly linearly with distance until the flow in (*b*) became too disturbed for it to be possible to follow the jet further. When the fluid in (*b*) was heavier than that in (*a*), there was a very different picture. Mixing began as before, but the rate of increase of depth soon became smaller and at a certain distance downstream, depending on the density difference and the rate of flow, the 'jet' region changed smoothly to one whose depth changed very little with distance. The turbulence in the mixed layer was damped out, and there was no further mixing with the salt solution below; the layer then remained of constant depth until it reached the second weir.

These conditions could be maintained only for a relatively short time. The loss of salt from the centre reservoir meant that the flow was not really steady, but, in any case, the flow was finally disturbed by a wave on the interface travelling back from the second barrier. However, it was found possible to make all the desired measurements during the few seconds between the establishment of a nearly steady state and the breakdown due to the passage of this wave.

In order to measure velocity, small plastic particles were introduced into the flow and photographed with a cinecamera. The particles, of density slightly less than that of fresh water so that they would rise slowly through it, were released when required from a device laid along the centre of the bottom of the tank; they were spaced at about 2 cm intervals over a distance of 60 cm. The films showed the particles rising through the salt solution, moving slightly backwards because of the back flow necessary to replace the salt removed by mixing, and then moving very quickly forward as they reached the lighter layer on top. The position of no horizontal flow was slightly below that where the sudden density change occurred. To extract numerical values for the velocities, the film was projected on to graph paper and the positions of the particles plotted frame by frame; the process is tedious, and only a few runs have been filmed and analysed completely in this way. However, the input velocity can readily be found from the flow rate and the depth at the weir; and it will be shown that useful information can be found using only this and the depth of the layer far downstream.

In the rather peculiar flow conditions of the tank, there is some doubt whether or not the velocity of the backflow should be considered a true flow in the ambient fluid; there is some suggestion that a circulation had been set up in the tank, in which case it should be ignored. In view of this doubt and the poor quality of the velocity profiles obtainable from the film, it was decided to compromise and analyse the experiments by taking the sum of one half the back flow

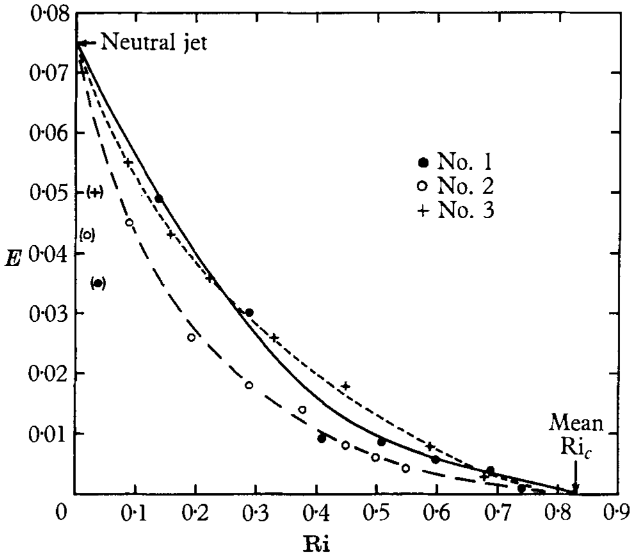

FIGURE 2. Entrainment, E, as a function of Richardson number, Ri, for three experiments on surface jets. The value of $E(0)$ has been taken from published results on neutral jets, and Ri_c obtained from a more extensive series of measurements of the depth of the final layer.

with the velocity near the top of the layer as V, and to assume that V_a was zero. For h an average of the depth of greatest density gradient and the depth of greatest velocity gradient was used. The individual points showed considerable scatter, but by averaging over a number of particles, smooth curves of $V(x)$ and $h(x)$ could be drawn. These functions were then inserted in the momentum equation (11) and it was usually found that they were not quite consistent; in that case, further small adjustments were made until (11) was satisfied. It was then possible to find E from (7) and Ri from (5). The results of three runs which have been fully analysed are shown in figure 2. The most important feature of the results is the rapid fall of E with increasing Ri.

It is of particular interest to know whether there is a 'critical' Richardson number, Ri_c, above which mixing is negligible; for example, in Expt. 1 it appears from figure 2 that Ri_c is about 0·74. A method of finding Ri_c is available which can be applied simply to a longer series of experiments. The momentum equation (11), with $S_1 = 1$, can be integrated and written in the form

$$\frac{h(1+\tfrac{1}{2}\mathrm{Ri})}{\mathrm{Ri}^{\frac{2}{3}}} = \text{constant}. \tag{15}$$

This shows that we need know only the input conditions and the ultimate depth of the layer in order to find Ri_c. This depth was determined from photographs of flows in which the fresh water was dyed. The mean of the values of $\mathrm{Ri}_c^{\frac{2}{3}}/(1+\frac{1}{2}\mathrm{Ri}_c)$ was evaluated from eleven experiments (since this quantity was much more nearly normally distributed than Ri_c itself), and from this we obtain a 'best' value of $\mathrm{Ri}_c = 0{\cdot}83 \pm 0{\cdot}10$. This value is marked on figure 2.

To sum up, we have found from this series of preliminary experiments that E falls off rapidly as Ri increases, and that for values of Ri greater than about 0·8 mixing is very small. A few words of caution are necessary before we attempt to compare these results with those of the later experiments. The numerical values of E and Ri depend on the definition of the velocities and depths; we used the velocity near the surface, which in this type of flow is likely to be close to but not identical with the mean velocity assumed in the development of the theory. In applying the theory we have also tacitly assumed that there is local equilibrium at any cross-section; this is clearly not so for points near the weir (as is shown by the low values of E obtained before the jet has become fully turbulent), and any later 'memory' effects will tend to overestimate E. Also it should be noticed that the smooth E *vs* Ri curves are a result of the method of analysis, and the difference between separate experiments, not the scatter on any one curve, should be used as an indication of the magnitude of random experimental errors.

6. The experimental determination of entrainment: inclined plumes

(*a*) *Earlier measurements*

The only previous measurements on inclined plumes known to us are those of Georgeson (1942). She liberated methane or hydrogen from a shallow box on the roof of a concrete gallery 5 ft. wide, 6ft. high and 100 ft. long, having a slope of 1 in 10, and measured the depth of the layer 25 ft. and 75 ft. from the source, and the velocity of the leading edge of the layer. Her interpretation was based on an arbitrary assumption about the rate of spread of the layer, but the results may be re-evaluated using the theory of §4. She found that the velocity was proportional to the cube root of the rate of inflow and varied little with x; this agrees with our equation (5) and our conclusions about a normal Richardson number provided we assume, as Georgeson did, that the velocity she measured was equal (or at least proportional) to the velocity of the layer. If we use this velocity to calculate Ri, and obtain E by (14) from the measured rate of growth of the layers (the depth grew from 8 in. at 25 ft. to 14 in. at 75 ft.), we find $E = 0{\cdot}01$ at $\mathrm{Ri} = 0{\cdot}37$. In view of the uncertainty of interpretation of these measurements, they are in good agreement with our results. A point representing these values is shown in figure 10.

More extensive experiments of the same type are currently being conducted by Leach (1959), and we are grateful to him for permission to quote his results before publication. He releases methane into a gallery having a slope of 1:25. This has a curved roof, so that the flow is not properly two-dimensional, but due allowance has been made for the effective width and depth of the layer in calculating entrainment. Leach measures methane concentration as a function of distance from the source and from the roof, and uses the continuity of density flux to define a mean velocity—a process which is equivalent to the use of equations (6) and (10) with $S_2 = 1$. An average of four experiments, with measurements at three cross-sections, leads to values of $E = 0{\cdot}0039$ and $\mathrm{Ri} = 0{\cdot}46$; these also are shown in figure 10. Note that Ri is increased and the point brought very nearly into agreement with our curve if S_2 is taken as 0·9, which we shall see is a plausible value at high Reynolds numbers.

(*b*) *Laboratory experiments*

(i) *The experimental channel*

In planning the laboratory experiments on inclined plumes emphasis was placed on the production and maintenance of a steady state. We required a nearly two-dimensional flow whose properties remained constant in time and which could be controlled over a wide range of flow rates, density differences and slopes. As in the surface jet experiments, salt solution and water were chosen as the working fluids in preference to gases since this enabled reasonable Reynolds numbers to be obtained with apparatus of practical size. (For a given Richardson number, Reynolds number and ratio of densities, the size of the apparatus is proportional to the two-thirds power of the kinematic viscosity, so if we had used air instead of water, the apparatus would have had to be six times as large.)

The channel which was used for most of the measurements was made as large as possible consistent with the available water supply. It consisted of a Perspex box $200 \times 60 \times 10$ cm, mounted with the largest faces vertical; it could be tilted to any angle between horizontal and vertical.

The channel was filled with fresh water, and salt solution supplied from a tank in the roof of the laboratory to an opening at the upper end of the bottom of the channel. The rate of flow was controlled by a series of nozzles which could be inserted into the supply pipe. The flow provided by each of these was measured, and so, knowing the density of the salt supply (determined by weighing), one could calculate the flux of density excess, A. The solution flowed in a 10 cm wide stream down the full length and was removed, together with the fresh water that it had entrained, from the bottom corner. There was a considerable body of water above the flowing layer, so the effect of the top of the channel was negligible; but, in order to maintain a steady state, fresh water had to be added to make up for that lost by entrainment. We knew from the theory that the velocity of flow and the entrainment at a fixed slope should be independent of distance, and hence that the inflow velocity should also be constant. Therefore extra water was supplied gently at a constant rate all along the top of the tank by forcing it through a long 'sock' about 10 cm in diameter made of cotton sheeting and connected at both ends through control valves to a constant head tank of fresh water. The arrangement is shown in figure 3.

Provision was made at 10 cm intervals along the centre of the bottom of the channel for the insertion of measuring instruments which could be traversed across the flow to give density and velocity profiles; since the flow was steady, there was sufficient time for such traverses to be made.

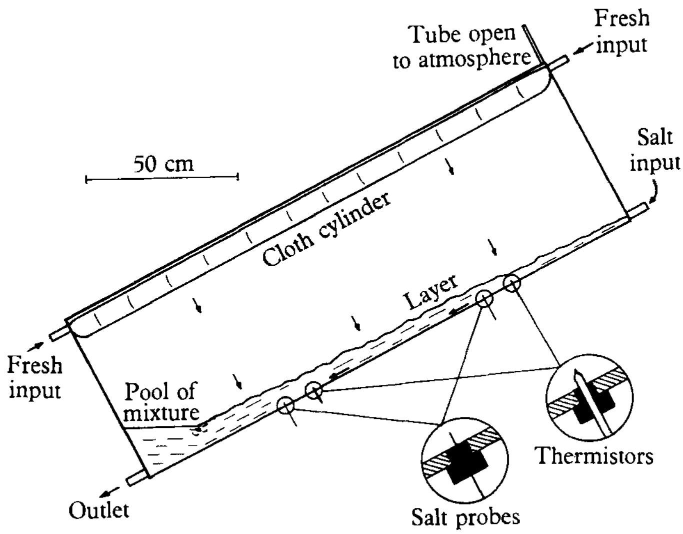

FIGURE 3. Sketch of the channel used to investigate inclined plumes.

The density was found by withdrawing samples from the desired position in the flow with a fine stainless steel tube and measuring salinity continuously in a simple conductivity cell. This consisted of two lengths of stainless steel tube (1 mm diameter for most of the experiments, but 3 mm diameter for experiments with the channel vertical when very weak samples had to be measured) mounted rigidly and joined by a short length of insulating polythene tubing leaving a 3 mm gap. A potential of 2·3 V, 50 c/s, was established across the cell and the current read on a meter; calibration was carried out directly against density using samples of known dilution.

The velocity profiles were obtained by using thermistors encased in the tips of glass tubes (Standard Telephones and Cables Ltd., Type F23) set transversely to the flow. Each of the thermistors, which were heated with a current of about 40 mA, 50 c/s, was placed in one arm of a Wheatstone bridge adjusted to balance when there was no flow. Movement of the water cooled the thermistor and threw the bridge off balance; the out-of-balance was indicated by a meter and used as a measure of the velocity at the tip of the thermistor. Calibration was carried out directly by placing the thermistors in a pipe where dyed patches of water could be timed between fixed marks. As with all instruments working on the 'hot-wire' principle, the output is highly non-linear (the instrument being most sensitive at low speeds) and depends on the temperature of the fluid, which was therefore recorded in each experiment and calibration run.

The averaging of the considerable fluctuations in the meter readings caused by the turbulence in the flow was done by eye, making a somewhat subjective allowance for the non-linearity of the instrument.

(ii) *Description of the flow*

The salt solution can be dyed so that it can be followed easily by eye. When this is done, a wide variety of phenomena can be directly observed, and it will be instructive to describe these briefly in qualitative terms before proceeding to the detailed results of the measurements.

When the flow is started, a rounded nose moves through the static water, usually overhanging its base slightly. At low slopes and large density differences this advances at about the mean flow rate of the layer behind it; it stays about the same depth as the layer apart from a small bump at the tip followed by a depression and grows only very slowly. At large slopes and smaller density differences the nose becomes very turbulent and builds up continually to many times the thickness of the layer. Its velocity is then much smaller than that of the layer behind. This fact must be kept in mind in interpreting measurements, such as those of Georgeson already discussed or the results for turbidity currents, in which only the nose velocity was determined (although in each of these cases it is in fact likely that the nose travelled at much the same speed as the layer). The properties of noses and the factors which determine their behaviour make an interesting study in themselves, but one which cannot be pursued here.

At low flow rates and small slopes, the steady layer behind the nose remains laminar with a sharp top and no mixing with the fluid above. If the flow rate is increased, waves appear on the interface, and increase in size until their amplitude is comparable with the depth of the layer. They eventually break and transfer material upwards into the overlying water. This is a mechanism of mixing which can operate in very stable conditions, even when the density gradients are so steep that no turbulence can persist.

At high flow rates and greater slopes, the whole of the salt layer becomes turbulent, and the layer spreads out as it entrains fresh water. The outer edge is an irregular succession of large eddies, but there is an inner region, a few millimetres thick, near the floor, which is less disturbed, and which appears as strongly coloured fluid with an irregular surface. This inner layer becomes less noticeable as the flow rate increases, but at intermediate Reynolds numbers it is very important. It appears from the measured profiles that the velocity maximum occurs near the position of steep density gradient; this means that the shear is zero near the top of the inner layer and so there is locally no source of turbulent energy at this level. It is not therefore surprising that there is little transfer across the interface. There is clearly strong interdependence of Reynolds number and Richardson number effects in the range of Reynolds numbers at which we can study inclined wall plumes in the laboratory; at high slopes (and hence low Ri) very low rates of flow give turbulent plumes with no appreciable inner layer, while at low slopes (and hence high Ri) much larger rates of flow are required for the plume to become fully turbulent. A quantitative discussion of the criterion for turbulence is beyond the scope of this paper, which must be confined to the turbulent case.

The nature of the entrainment process and the effectiveness of our method of adding fresh water to the channel may be demonstrated as follows. Dye is added briefly to the fresh water supply; this seeps out through the sock and travels straight across the channel perpendicular to the plume at all slopes. At the edge of the plume it is trapped by the large eddies and, as it becomes part of the plume fluid, travels at right angles to its previous direction and much faster. None of the fluid in the plume escapes: transfer is always into the turbulent fluid.

If too little fresh water is supplied, a pool of dilute salt solution builds up at the bottom of the tank. Usually, the supply of fresh water is adjusted until this pool just disappears, but for some purposes it is desirable to retain it; its position may be used as a measure of the entrainment as described later.

In the following sections three approaches to the problem of measuring entrainment as a function of Richardson number are described. We started with measurements involving the recording and integration of velocity profiles, but as our understanding of the process increased, proceeded to use simpler and more effective methods of characterizing the flow.

(iii) *Measurements of profiles*

In figure 4 are shown typical mean profiles of velocity and density difference for an inclined plume with a slope of 14°. Note that the velocity profiles have a steep increase to the maximum and then a gentle, nearly linear decrease to zero at the edge; the salt profiles show a very concentrated layer near the wall, but the concentration falls off sharply at about the height of the velocity maximum. The edge of the measurable salt is close to the edge of the layer

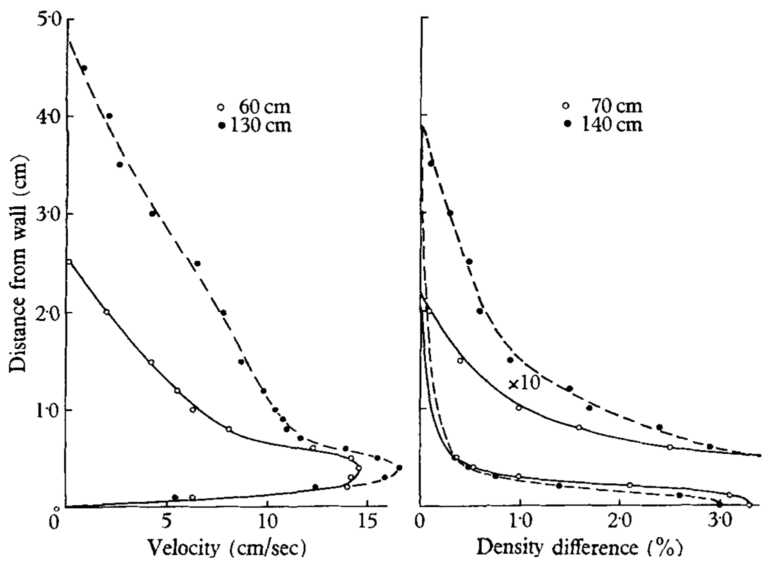

FIGURE 4. Typical profiles of (*a*) velocity, and (*b*) density in an inclined plume. 14°; $A = 144$.

indicated by the velocity profiles. The dilution of the plume and the increase of its depth with distance downstream are clearly shown.

We have integrated profiles such as these to obtain V and h defined by (1) and (2) and have then calculated E and Ri using (4) and (5). The process is tedious, but the results for a series of experiments covering a wide range of conditions are presented in figure 5. It is clear that we do not obtain by this means a single value of Ri for each slope or even a single curve of E *vs* Ri. Part of the difficulty is the limited accuracy of the velocity measurements (probably not better than $\pm 10\%$) and the fact that the measured velocities have to be cubed to obtain Ri. There is, however, a trend in the results suggesting that E is lower at higher Reynolds numbers (the numbers marked on the points are

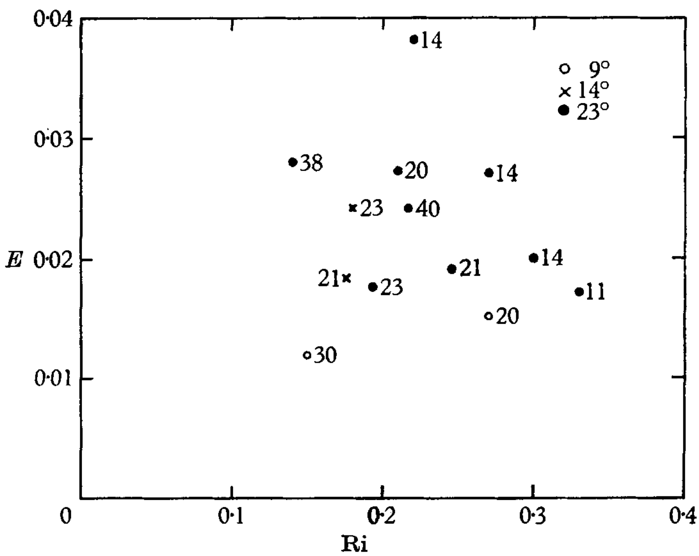

FIGURE 5. Entrainment into plumes as a function of Richardson number, obtained by integration from measured profiles. The numbers marked on the points represent the mean value of Vh during the runs.

values of Vh in the middle of the length over which E was measured). This odd effect may be attributed to the inner concentrated layer of salt. At low Reynolds numbers this carries most of the salt flux; little salt can pass the velocity maximum, but there is a turbulent region above it which is therefore hardly stabilized at all by the salinity gradient. At higher Reynolds numbers the salt is spread right across the moving layer and has a stronger inhibiting action on entrainment. This view is supported by a calculation of the factor S_2 defined by (10), which gives a measure of the spread of the salt relative to the velocity profile. We find that lower entrainment is associated with higher values of S_2, the range of S_2 for the observations shown in figure 5 being from 0·6 in a flow at low Reynolds number at 23° to 0·9 at the highest Reynolds number obtainable at 9°.

The same integration of salinity profiles enables us to estimate the friction coefficient C from (13). This is found to take values greater than would be found if the region below the velocity maximum were laminar but less than that associated with pipe flow in a uniform fluid. The measurements are not sufficiently accurate for it to be possible to describe how the friction coefficient depends on stability and Reynolds number. Also, when the layer has thickened, its depth is no longer negligible compared with its width, and friction on the side wall is an additional complication.

(iv) *Analysis of the outer parts of the layer*

The full analysis of the profiles gave very little useful information after a great deal of labour. We shall now concentrate on the region beyond the velocity maximum, and present a quicker method of analysis. Instead of measuring V and h by integrating the profiles, we choose a characteristic velocity and

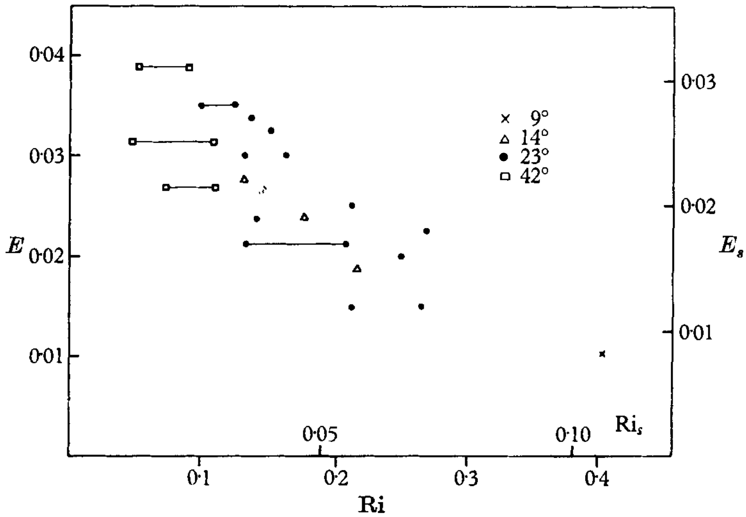

FIGURE 6. Entrainment as a function of Richardson number based on measurements of the outer parts of the plume. Scales are shown for the directly measured E_s and Ri_s, and also for the equivalent E and Ri which relate to mean properties.

length which may be found by inspection in each case. We note that the outer parts of the velocity profiles are nearly linear and so we use the intercepts of this straight portion on the height axis and on a parallel to the velocity axis through the velocity maximum, to define h_s and V_s; the density at a standard distance above the velocity maximum, $0{\cdot}2h_s$, is used to define a flux $A_s = \Delta_{\frac{1}{5}} h_s V_s$. A single integral over each velocity profile is used to determine $d(Vh)/dx$ and hence E.

The relations between V_s and V, and Ri_s and Ri, were found from a study of those cases for which both types of analysis were available to be $V = 0{\cdot}80 V_s$ and $\mathrm{Ri} = 3{\cdot}8\mathrm{Ri}_s$. It should be noted, however, that these conversion factors are based almost entirely on experiments at 23° and there is some uncertainty in applying them to higher slopes; and in fact at the lower Richardson numbers the results of this method do not agree well with those of our other methods (figure 10).

The results of the simplified analysis are plotted in figure 6, where scales for both Ri and Ri_s and for E and E_s are shown. Although there is still a considerable scatter, it is less than in figure 5 and a detailed examination shows no trend with Reynolds number. Note that sometimes two values of Ri are shown for the same E; these represent occasions on which the flow was accelerating slightly between the two measuring positions. There is clearly a decrease of entrainment with increasing Richardson number and an increase of normal Ri with decreasing slope.

(v) *Direct measurement of velocity and entrainment as a function of slope*

The success of the simple method of analysis used for the outer parts of the plume in the last section encouraged us to simplify even further and to seek a definition of a characteristic velocity which could be measured without the need to obtain profiles at all.

We have noted that the velocity maximum occurs close to the region of large density changes, which with coloured salt solution is visible as a sharp interface a few millimetres from the wall. Irregularities or turbulent eddies at this level may be followed by eye and their passage between fixed marks on the channel timed to give a reproducible measure of velocity. Comparison of velocities obtained in this way with measured profiles shows that, when a sharp layer is present, one tends to follow waves on the interface and find a velocity slightly larger than the maximum mean particle velocity. The relation between V and the layer velocity, V_L, is $V = 0{\cdot}60 V_L$ at 23°; at higher slopes, when there is no sharp inner layer, there is evidence that V_L is close to the maximum velocity and that $V = 0{\cdot}7 V_L$. (Note that if V_L was exactly the maximum velocity, the factor would be 0·70 for Gaussian profiles and 0·67 for triangular profiles.) The results have been interpreted on the assumption that the factor is a linear function of slope varying from 0·60 at 23° to 0·70 at 90°. Any uncertainty introduced by this somewhat arbitrary procedure is not large. Richardson numbers have been calculated using A found from the known input rates and densities of the salt supply.

The entrainment was measured by making use of the pool of dyed salt solution at the bottom corner of the channel. When the flow rates are adjusted so that a steady small pool forms, we know that the length of the entraining layer extends from the input to the top surface of this pool; mixing within the pool does not remove any further fresh water from the channel. The total rate of output was measured with a bucket and a stop-watch, and so, after subtracting the salt input, $d(Vh)/dx$ could be found. Thus with the aid of the velocities obtained from the timing of the layer we can evaluate E. The method is much less laborious than the earlier ones, and a more extensive series of measurements has been possible.

The results, which cover a wide range of output rates, density differences and slopes, are shown in figure 7. We have also plotted E and A/V^3 as functions of slope in figures 8 and 9, since these relations are often required in practice. It is of particular interest to note that the value of E found for a vertical wall plume is 0·087, compared with the value of 0·075 for a neutral jet. We would not attach much significance to the difference between these values; but we can say confidently that E is not as high as 0·22 which, as mentioned earlier, is the value corresponding to the measurements of Rouse *et al.* (1952).

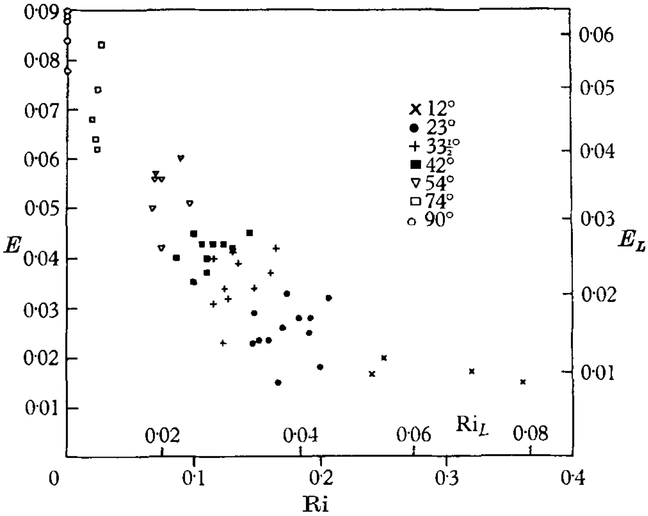

FIGURE 7. Entrainment as a function of Richardson number for inclined plumes, using the direct visual measurements of velocity and entrainment. The scales E_L and Ri_L relate to the 'layer velocity' and E and Ri to the mean velocity.

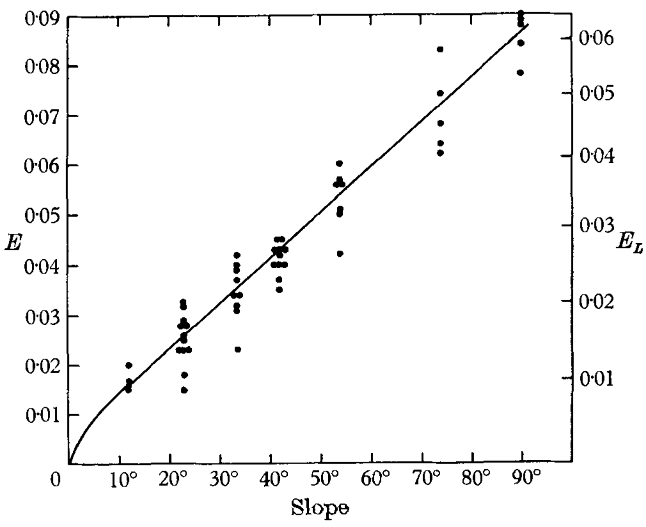

FIGURE 8. Measurements of entrainment into an inclined plume as a function of slope. Scales show both E_L based on the 'layer velocity' and E based on the equivalent mean velocity.

(vi) *Comparison of the several methods*

In figure 10 we have collected together the mean curves of E *vs* Ri found by the three methods: the surface jet, the profile measurements analysed using the outer parts of the plume and the timing of the layers. Agreement is good, particularly when it is remembered how sensitive the values of Ri are to the velocity measurements. The surface jet tends to give the highest values of E for a given Ri; if this is regarded as significant, it might be explained as a 'memory' effect arising because the jet is not exactly in local equilibrium. It is likely that the Richardson numbers taken from the profile measurements are systematically too low at small values of Ri, as mentioned in (iv) above.

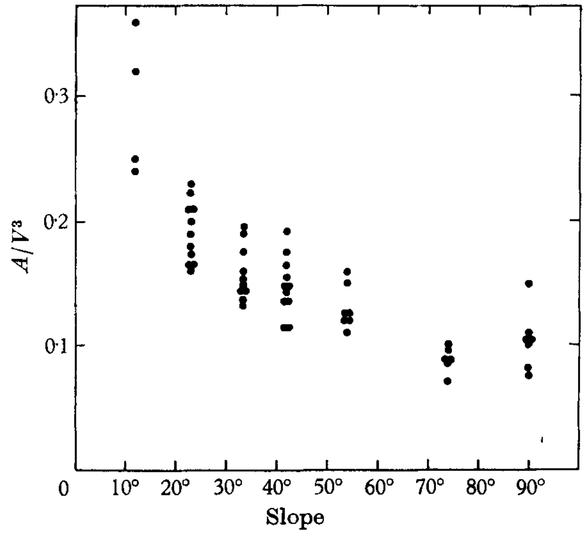

FIGURE 9. Measurements of A/V^3 as a function of slope. The points have been obtained by the 'layer velocity' method but are plotted using the equivalent mean velocities.

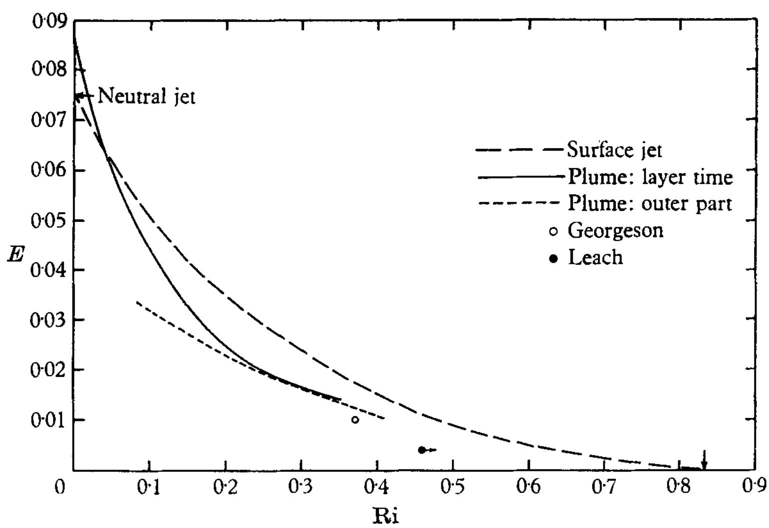

FIGURE 10. Comparison of the three series of measurements of entrainment as a function of Richardson number, all based on equivalent mean velocities. Also shown are points calculated from the measurements of Georgeson and Leach.

The curves are less firmly based at high values of Ri than elsewhere. This is a pity since important flows occur at low slopes, but in order to obtain satisfactory measurements in this region, a much larger scale experiment than is practical in the laboratory seems to be required. However, as we shall see in the next section, when the ambient fluid is in motion the relevant Richardson numbers are smaller and the region which we have examined in detail is important even at low slopes.

7. The application of the results

(a) Prediction of mean velocity and concentration

The laboratory results may be used confidently to make predictions of the velocity of flow of a larger scale turbulent density current on a steep slope. All that is required is an estimate of the rate of output of light or heavy fluid; small changes in Ri due to Reynolds number effects or changes in the friction coefficient will be unimportant since V is proportional to $\mathrm{Ri}^{\frac{1}{3}}$. This point is brought out clearly in figure 11, in which we show measurements of V based on profiles recorded at a slope of 23°, plotted against A on logarithmic scales; V is proportional to $A^{\frac{1}{3}}$ over a wide range of A and the relation should continue to hold at higher values. The constant of proportionality could have been obtained from figure 9.

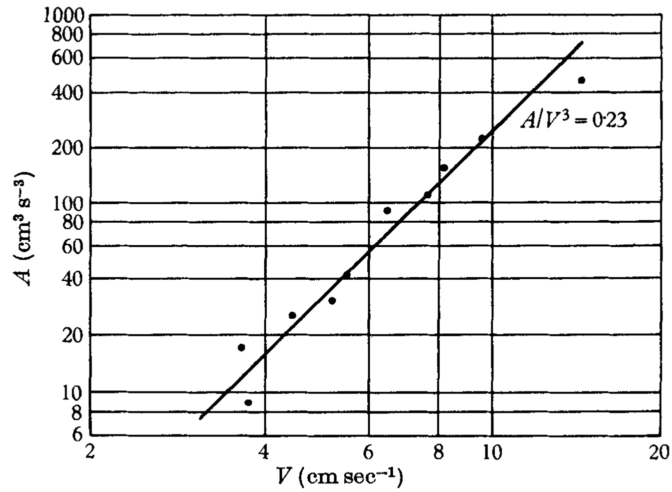

FIGURE 11. Measurements of the flux of density difference, A, as a function of the velocity V at 23°, plotted on logarithmic scales.

At low slopes, our results suggest that E becomes very small, so that even at large Reynolds numbers there should be negligible mixing of a current with its surroundings and so little friction at the interface. The weight of the layer must then be balanced solely by friction on the floor and

$$S_2 \frac{A}{V^3} \sin \alpha = C. \tag{16}$$

This result is important in the theory of turbidity currents (Charnock 1959).

At a given slope and rate of emission and hence a fixed V, the concentration in a plume at high Reynolds number is inversely proportional to h. Since $dh/dx = E$, the dilution can readily be calculated using the results for E as a function of slope (figure 8).

(*b*) *The effect of motion of the ambient fluid*

There is considerable interest in the question of how a steady velocity in the ambient fluid affects the flow of a wall plume. The meteorologist wishes to know how large an upper wind is necessary to destroy a katabatic flow at the surface; and the mining engineer is interested in how fast a downhill ventilating flow is needed to prevent a methane layer flowing uphill along the roof of a sloping mine roadway.

Using the measurements of E *vs* Ri, we can now supply rough answers to these questions. The results are not very sensitive to small changes in this relation, and are probably adequate even if there is a tendency for an opposing flow of ambient fluid to increase the apparent value of E (as might perhaps be inferred from a remark of Townsend (1956)).

Let us again consider the case where there is a line source of density difference of strength A, though this is a considerable simplification of the meteorological situation where cooling may extend over tens of kilometres. The equations required are

$$\Delta(V+V_a)\,h = A = \text{constant}, \tag{3}$$

$$\frac{dh(V+V_a)}{dx} = E\,|V|, \tag{4}$$

where E is a function of the Richardson number which is given by

$$\mathrm{Ri} = \frac{h\Delta\cos\alpha}{V^2} = \frac{A\cos\alpha}{V^2(V+V_a)}; \tag{5}$$

and the momentum equation, which in this case may be written

$$\frac{d(V+V_a)\,Vh}{dx} = -(V+V_a)^2\,C\,\mathrm{sgn}\,(V+V_a) + S_2\Delta h\sin\alpha - \tfrac{1}{2}\cos\alpha\,\frac{d\Delta h^2 S_1}{dx}. \tag{17}$$

These equations will be valid only when V is positive, that is, when the motion of the ambient fluid is either opposed to the natural motion of the layer or, if in the same direction, is less than the speed at which the layer is travelling. In the case where the ambient fluid is dragging the layer forward in the same direction as gravity with friction as the only retarding force, the profiles will be very different from those assumed for plumes and our values of E are unlikely to apply. The velocity of the layer itself, $V+V_a$, can take either sign.

In the normal state in which $dV/dx = 0$, the equations lead to

$$\left(1+\tfrac{1}{2}S_1\mathrm{Ri}\,\frac{V}{V+V_a}\right)E = S_2\mathrm{Ri}\tan\alpha - C\,\frac{(V+V_a)^2}{V^2}\,\mathrm{sgn}\,(V+V_a). \tag{18}$$

For given values of α and C, and using our results for $E(\mathrm{Ri})$, the equation (18) can be solved numerically to give $(V+V_a)/A^{\frac{1}{3}}$ as a function of $V_a/A^{\frac{1}{3}}$, that is, the non-dimensional layer velocity as a function of that in the ambient fluid. The calculation has been carried out using $S_1 = 0{\cdot}25$ and $S_2 = 0{\cdot}75$, for $\alpha = 0{\cdot}02$ and $\alpha = 0{\cdot}1$, with $C = 0$ and $C = 0{\cdot}02$. The latter friction coefficient is a high value which is unlikely to be reached except in small-scale laboratory experiments. The results are shown in figure 12. It will be seen that when V_a opposes the natural flow and is greater than a value ranging between $2{\cdot}4A^{\frac{1}{3}}$ when $C = 0$, $\alpha = 0{\cdot}02$, and $3{\cdot}2A^{\frac{1}{3}}$ when $C = 0{\cdot}02$, $\alpha = 0{\cdot}1$, it is possible for the layer to flow in either direction. Three values of $(V+V_a)$ satisfy (18), but it is fairly clear that the middle one is unstable and that the layer can either flow very slowly with gravity, concentrating the available density difference so as to achieve the

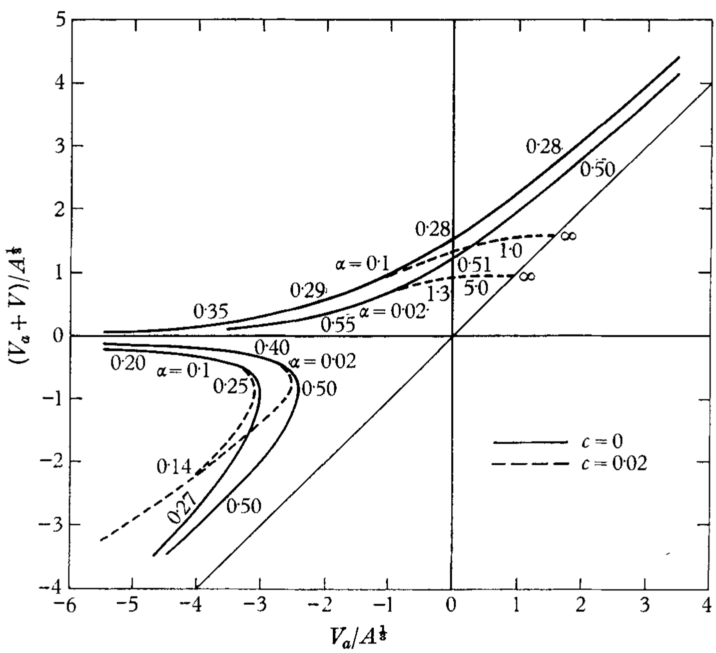

FIGURE 12. Calculations of the velocity of an inclined plume with a moving ambient stream, based on the experimental $E(\mathrm{Ri})$ relation shown in figure 10. $V+V_a$ is the velocity of the layer, V_a that of the ambient stream. Curves are shown for two values of the slope and two friction coefficients; the velocity necessary to reverse the direction of flow is seen to be relatively insensitive to changes in these parameters.

necessary Richardson number, or at a moderate speed in the reverse direction. Physical intuition urges that since only a small fluctuation in speed from the slow forward moving state is needed to take the layer to the unstable solution, it is likely that in practice the reverse flow will be realized whenever it can exist.

The curves of figure 12 for different slopes and friction coefficients lie close together, showing that the velocity of ambient fluid required to produce reversal, like the free forward velocity, varies only slowly with these parameters and all that is needed to predict it is a good estimate of A. However, the Richardson number, values of which are marked on figure 12, does depend on the slope, and the corresponding values of E, which is a strong function of Ri, vary more markedly. We note, as mentioned previously, that small values of Ri are relevant when the ambient fluid is acting against gravity; and for a given ambient velocity there is much more mixing when the slope is steep. The entrainment becomes very small in the case of moderate ambient flows assisting gravity.

Although these conclusions and the numerical values are strictly applicable only to the case of a turbulent layer and laminar ambient stream, it is unlikely that they will be affected greatly by a low level of turbulence in the ambient flow. It is not possible for us to test the theory in our large channel because of the amount of water which would be required to produce the ambient stream, but we have begun some experiments in a small tiltable channel 10 cm wide by

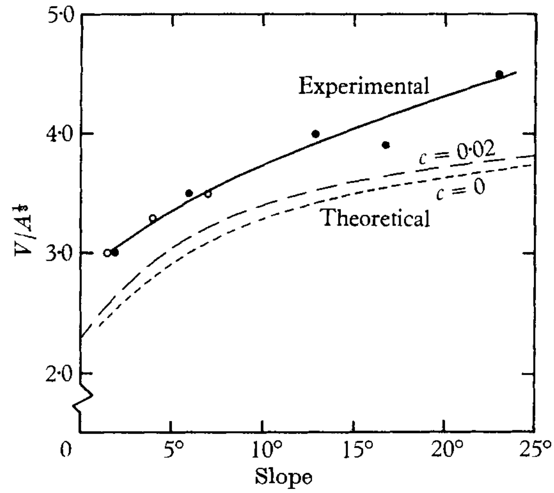

FIGURE 13. Experimental values of the reversal velocity as a function of slope, compared with calculations carried out for two values of friction coefficient. Dots: $A = 21$. Circles: $A = 63$.

5 cm deep and 250 cm long, which are of interest in this connexion. Salt solution is released from a slit across the bottom of the channel near the centre of its length, and may travel either with or against the main stream (which in this channel is necessarily turbulent at moderate velocities), depending on the rate of output, the main stream velocity and the slope. In figure 13 are shown the measured mean velocities of the main flow necessary to produce reversal of the layer for a range of slopes. These are compared with the theoretical values, calculated on the basis of diagrams like figure 12, for two values of friction coefficient. The agreement is satisfactory and the measurements demonstrate the small dependence on slope. Note particularly that flow against the main stream is possible at very small slopes.

(c) *Typical values of A*

Finally we may give some estimates of typical values of A in practical cases, from which velocities may be calculated.

For a katabatic wind, suppose there is a line source equivalent to an area 50 km long in the direction of greatest slope which is cooled at 0·1 cal cm^{-2} min^{-1}: then $A = 10^8$ cm^3 sec^{-3}. The velocity of free flow would be about

$$600 \text{ cm sec}^{-1} = 12 \text{ knots},$$

and the velocity of upper wind necessary to reverse it is

$$2{\cdot}6A^{\frac{1}{3}} = 1200 \text{ cm sec}^{-1} = 24 \text{ knots}.$$

These values are not unreasonable. It is hoped in a later investigation to compare the theory with meteorological observations, and for this purpose it may be necessary to extend the theory slightly to take account of the Coriolis force due to the earth's rotation.

As an example of the release of methane in a mine gallery, let us suppose that 10 ft.^{3} min^{-1} is released in a roadway 5 ft. wide. A is then 10^4 cm^3 sec^{-3} and the ventilating flow needed to reverse the layer would be 57 cm sec^{-1} = 120 ft. min^{-1}.

In the case of turbidity currents, our results may be used in reverse. There are well-documented cases in which turbidity currents have broken a succession of telegraph cables on the ocean floor and so their front velocity is well known. In this case E is very small so it is only necessary to guess C. For example, in the Grand Banks turbidity current of 1929, velocities of 55 knots were reported, so with $C = 0{\cdot}003$, we find $A = 6\times 10^{10}$ cm^3 sec^{-3}.

8. Conclusion

The assumption that the rate of entrainment into a turbulent inclined plume is a function of the overall Richardson number has led to a consistent picture of such flows. Our experiments have determined the empirical function $E(\mathrm{Ri})$ with an accuracy sufficient for most applications except in the range of high Richardson numbers where E is very small. Here much larger scale experiments are desirable.

It is not claimed that this approach is in any sense fundamental, but it forms a natural extension of an idea which has been used successfully in uniform fluids. The experiments could, perhaps, be refined further to investigate systematically the effects of Reynolds number (for example, to explain the details of figure 5). This seems to be taking the 'overall' approach too far, however; it is likely to be more profitable to look at the local rates of transfer of salt and momentum and to try to relate them to local stability parameters at different parts of the flow. Work is now being continued along these lines, both for simple wall plumes and (in the smaller channel mentioned in the last section) for a flow with an opposing turbulent ambient fluid.

9. Acknowledgements

The first part of the experimental work was carried out in the Cavendish Laboratory while the second author held an Overseas Scholarship from the Royal Commission for the Exhibition of 1851. The second author also wishes to acknowledge the financial support of the Ministry of Power during the period of the later work in Manchester.

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